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CHARACTERIZATIONS OF SEMI COMPACT SPACES

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CHARACTERIZATIONS OF SEMI COMPACT SPACES

by

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A Dissertation Presented in Partial Fulfillment
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ABSTRACT

This dissertation consists of characterizing semi compact spaces in general and characterizing semi compact T_0 spaces, T_1 spaces, and R_0 spaces in particular.

In Chapter II it is shown that the following are equivalent:

- (a) (X, T) is semi compact,
- (b) for each $Y \in T$, (Y, T_Y) is semi compact,
- (c) for each $O \in T$, $\text{Fr}(O)$ is finite and if $\{O_n\}_{n \in \mathbb{N}}$ is a sequence of open sets in X , then there exists $m \in \mathbb{N}$ such that $O_p \subset \bigcup_{i=1}^m O_i$ for all $p \in \mathbb{N}$,
- (d) for each $O \in T$, $\text{Fr}(O)$ is finite and every chain in T , partially ordered by set containment, has a maximal element,
- (e) for each $O \in SO(X, T)$, $\text{Fr}(O)$ is finite and every chain in $SO(X, T)$, partially ordered by set containment, has a maximal element,
- (f) for each $O \in SO(X, T)$, $\text{Fr}(O)$ is finite and if $\{O_n\}_{n \in \mathbb{N}}$ is a sequence of semi open sets in X , then there exists $m \in \mathbb{N}$ such that $O_p \subset \bigcup_{i=1}^m O_i$ for all $p \in \mathbb{N}$,
- (g) for each $Y \subset X$, (Y, T_Y) is semi compact,
- (h) for each $Y \subset X$, every cover of Y by sets semi open in X has a finite sub-cover,
- (i) (X, T) is type 1, type 2, type 3, or the finite topological sum of nonempty disjoint spaces that are type 1, type 2, or type 3,
- (j) every semi homeomorphic image of (X, T) is semi compact, and
- (k) (X, T) can be semi embedded in a semi compact space.

In Chapter III it is shown that (X,T) is semi compact T_1 if and only if (X,T) is type (1,1), type (2,1), type (3,1), or a finite topological sum of nonempty disjoint spaces that are type (1,1), type (2,1), or type (3,1), and that two equivalent formulations of semi compact R_0 spaces are (a) $(X_0, Q(X_0))$ is semi compact T_1 and for each closed set C in X_0 with empty interior, each element of C is a finite set, and (b) (X,T) is type (1,0), type (2,0), type (3,0), or a finite topological sum of nonempty disjoint spaces that are type (1,0), type (2,0), or type (3,0).

It is further shown in Chapter III that 1st countability and 2nd countability are equivalent in semi compact R_0 and T_1 spaces.

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CHAPTER I

INTRODUCTION

What would happen to the familiar concepts of continuity, convergence, compactness, etc., if open sets were generalized to open sets unioned with portions of their boundaries? Could topologies be described by means of these extended sets? Questions of this nature led to the definition by Norman Levine in 1963 of semi open sets.

Definition 1.1. A set A in a topological space (X, T) is semi open if and only if there exists $0 \in T$ such that $0 \subset A \subset \bar{0}$ (where $\bar{0}$ is the closure of 0 in X) [10].

Open sets are semi open, but semi open sets are not necessarily open. This may be seen by considering R with the usual topology, where $(a, b]$ is semi open but not open.

Semi open sets have been used to define and investigate many new properties and concepts in topology. Among these are semi closure, semi continuity, semi convergence, and semi- T_i spaces, $i = 0, 1, 2$.

Throughout this paper, $SO(X, T)$ will denote the collection of sets semi open in X , N will denote the set of positive integers, and $Fr(0)$, the frontier of 0 , will denote the set of boundary points of the set 0 .

Insisting that the inverse image of every open set be semi open is not enough to imply continuity of a function from one topological space to another. Let $[0, 1]$ have the usual topology and define $f : [0, 1] \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{if } 1/2 < x \leq 1 \end{cases}.$$

The inverse image of every open set is semi open, but f is not continuous. This situation led to the definition by Levine in 1970 of semi continuity.

Definition 1.2. Let (X,T) and (Y,S) be topological spaces. Then $f : (X,T) \rightarrow (Y,S)$ is semi continuous if and only if $f^{-1}(0)$ is semi open in X for each $0 \in S$ [10].

Insisting that every semi open cover can be replaced by a finite sub-cover is enough to imply compactness, but compactness is not enough to imply that every semi open cover can be replaced by a finite sub-cover. Let R^2 be the plane with the usual topology, let $A = (0,1] \times [0,1]$, and let $B = [0,1] \times [0,1]$. B is compact, but the semi open cover $\{(x,y) \cup A \mid x = 0, 0 \leq y \leq 1\}$ has no finite sub-cover. In 1982 Dorsett defined semi compactness.

Definition 1.3. Let (X,T) be a topological space and let $A \subset X$. Then (X,T) is semi compact if and only if every semi open cover of X , i.e., a cover of X by sets semi open in X , has a finite sub-cover, and A is semi compact if and only if every cover of A by sets semi open in X has a finite sub-cover [6].

Study was then directed toward characterizations of semi compactness. Dorsett showed in 1981 that a space (X,T) is semi compact if and only if for each $0 \in T$, $\text{Fr}(0)$ is finite, and there does not exist an infinite sequence of nonempty disjoint open sets [6]. Others are given in the characterization theorem of Chapter II.

Other research efforts were directed toward describing semi compact T_2 spaces and R_1 spaces. Dorsett showed in 1980 that a

T_2 space (X, T) is semi compact if and only if X is finite, and that one equivalent condition for a semi compact R_1 space (X, T) is that $\{\{\bar{x}\} | x \in X\}$ is a finite subset of T [5]. In Chapter III characterizations are given for semi compactness in T_0 spaces, T_1 spaces, and R_0 spaces. Also, 1st countability is shown to be equivalent to 2nd countability in semi compact T_1 spaces and R_0 spaces.

Previous definitions and known results utilized in the proofs presented in this paper are listed below.

Definition 1.4. If $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$ is a family of nonempty disjoint topological spaces, $X = \bigcup_{\alpha \in A} X_\alpha$, and $T = \{G \subset X | (G \cap X_\alpha) \in T_\alpha \text{ for all } \alpha \in A\}$, then (X, T) is called the topological sum of $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$ [13].

Theorem 1.1. Let (X, T) be a topological space. If $S \in SO(X, T)$ and $0 \in T$, then $S \cap 0 \in SO(X, T)$ [11].

Definition 1.5. A one-to-one function f from a topological space (X, T) onto a topological space (Y, S) is a semi homeomorphism if and only if $f(0) \in SO(Y, S)$ for all $0 \in SO(X, T)$ and $f^{-1}(U) \in SO(X, T)$ for all $U \in SO(Y, S)$ [2].

Definition 1.6. Let (X, T) be a topological space and let R be the equivalence relation on X defined by xRy if and only if $\{\bar{x}\} = \{\bar{y}\}$. Then the T_0 -identification space of (X, T) is $(X_0, Q(X_0))$, where X_0 is the set of equivalence classes of R and $Q(X_0)$ is the decomposition topology on X_0 , which is T_0 [13].

Theorem 1.2. The natural map $P : (X, T) \rightarrow (X_0, Q(X_0))$ is continuous, closed, open, onto, $P^{-1}(P(0)) = 0$ for all $0 \in T$, $P^{-1}(\text{Fr}(0)) = \text{Fr}(P^{-1}(0))$ for all $0 \in Q(X_0)$, and $P^{-1}(C)$ is connected if and only if C is connected [7].

Definition 1.7. Let (X,T) and (Y,S) be topological spaces and let $f : (X,T) \rightarrow (Y,S)$. Then f is almost-open if and only if for each $0 \in S$, $f^{-1}(\overline{0}) = \overline{f^{-1}(0)}$ [12].

Theorem 1.3. Let (X,T) and (Y,S) be topological spaces. If $f : (X,T) \rightarrow (Y,S)$ is semi continuous, almost-open, and onto, and (X,T) is semi compact, then (Y,S) is semi compact [6].

Theorem 1.4. Let (X,T) be a topological space. Then the following are equivalent:

- (a) (X,T) is R_0 ,
- (b) $(X_0, Q(X_0))$ is T_1 ,
- (c) $X_0 = \{\{\overline{x}\} | x \in X\}$, and
- (d) $\{\{\overline{x}\} | x \in X\}$ is a decomposition of X [8].

Theorem 1.5. If the topological space (X,T) is semi compact, then there does not exist an infinite closed set with empty interior [4].

Theorem 1.6. Let $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$ be a collection of nonempty disjoint topological spaces and let (X,T) be the topological sum of $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$. Then (X,T) is R_0 if and only if (X_α, T_α) is R_0 for all $\alpha \in A$ [9].

Theorem 1.7. A topological space (X,T) is 2nd countable if and only if its T_0 -identification space is 2nd countable [8].

Theorem 1.8. Let (X,T) be a topological space and let $Y \subset X$. Then $Y \in SO(X,T)$ if and only if for each $A \in SO(Y, T_Y)$, $A \in SO(X,T)$ [11].

CHAPTER II

CHARACTERIZATIONS OF SEMI COMPACT SPACES

The fact that there are many interesting and useful characterizations of compactness in general topology led to the question of whether or not there are characterizations of semi compactness other than Dorsett's result mentioned previously.

At the same time, the fact that in general topology it has been shown that every topological space can be embedded in a compact space led to the question of whether or not every topological space can be embedded in a semi compact space. Investigation of this problem resulted in the discovery that every subspace of a semi compact space is semi compact. This result implies not only that only semi compact spaces can be embedded in a semi compact space, but also that an equivalent formulation of semi compactness is that every subspace be semi compact. This result is included in the characterization theorem of this chapter.

Theorems 2.1 and 2.2 are necessary for the proof of the characterization theorem, Theorem 2.3.

Theorem 2.1. If (X, T) is semi compact, then (a) X is finite, or (b) there exists $U \in T$ such that U is infinite and if $V \in T - \{\emptyset\}$ such that $V \subset U$, then $U \subset \overline{V}$.

Proof: Suppose X is infinite and there does not exist $U \in T$ such that U is infinite and if $V \in T - \{\emptyset\}$ such that $V \subset U$, then $U \subset \overline{V}$. Let $N_p = \{n \in \mathbb{N} \mid n \leq p\}$. For $p \in \mathbb{N}$ there exists $f_p : N_p \rightarrow (T - \{\emptyset\}, T - \{\emptyset\})$, where

$f_p(i) = (O_i, Q_i)$, such that (1) if $i \in N_p - \{1\}$, then f_i is an extension of f_{i-1} and $O_i \subset X - \bigcup_{j < i} O_j$, (2) $Q_i \subset X - \bigcup_{j \leq i} O_j$ for all $i \in N$, and (3) Q_i is infinite for all $i \in N$. The proof is by mathematical induction.

Since $X \in T$ and X is infinite, then there exists $V \in T - \{\emptyset\}$ such that $V \subset X$ and $X \not\subset \bar{V}$. Then $X - \bar{V}$ is open and nonempty. Since $X = V \cup \text{Fr}(V) \cup (X - \bar{V})$ and $\text{Fr}(V)$ is finite, then either V or $X - \bar{V}$ is infinite. If V is infinite, let $O_1 = X - \bar{V}$ and $Q_1 = V$; otherwise, let $O_1 = V$ and $Q_1 = X - \bar{V}$. Then $f_1 : N_1 \rightarrow (T - \{\emptyset\}, T - \{\emptyset\})$ defined by $f_1(1) = (O_1, Q_1)$ satisfies the desired conditions. Assume the statement is true for $p = k$. Since Q_k is open and infinite, then there exists $W \in T - \{\emptyset\}$ such that $W \subset Q_k$ and $Q_k \not\subset \bar{W}$. Thus $Q_k \cap (X - \bar{W})$ is open and nonempty. Since $Q_k = W \cup (\text{Fr}(W) \cap Q_k) \cup (Q_k - (X - \bar{W}))$ and $\text{Fr}(W)$ is finite, then either W or $Q_k - (X - \bar{W})$ is infinite. If W is infinite, let $O_{k+1} = Q_k - (X - \bar{W})$ and $Q_{k+1} = W$; otherwise, let $O_{k+1} = W$ and $Q_{k+1} = Q_k - (X - \bar{W})$. Since $W \subset Q_k$ and $Q_k \subset X - \bigcup_{j \leq k} O_j$, then both W and Q_k are contained in $X - \bigcup_{j \leq k+1} O_j$. Also, $Q_{k+1} \cap O_{k+1} = \emptyset$. Thus, $O_{k+1} \subset X - \bigcup_{j < k+1} O_j$, $Q_{k+1} \subset X - \bigcup_{j \leq k} O_j$, and Q_{k+1} is infinite. Then $f_{k+1} : N_{k+1} \rightarrow (T - \{\emptyset\}, T - \{\emptyset\})$ defined by

$$f_{k+1}(i) = \begin{cases} (O_{k+1}, Q_{k+1}) & \text{if } i = k+1 \\ f_k(i) & \text{if } i \leq k \end{cases}$$

has the desired properties. Hence the statement is true for each $p \in N$, and there exists an infinite sequence of nonempty disjoint open sets,

which is a contradiction.

Theorem 2.2. If $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$ is a family of nonempty disjoint topological spaces, then the topological sum (X, T) of $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$ is semi compact if and only if A is finite and (X_α, T_α) is semi compact for all $\alpha \in A$.

Proof: Suppose (X, T) is semi compact. If A is not finite, then $\{X_\alpha\}_{\alpha \in A}$ is a semi open cover of X with no finite sub-cover. Suppose (X_ψ, T_ψ) is not semi compact for some $\psi \in A$. Then there is an infinite collection of semi open sets $\{S_\beta\}_{\beta \in B}$ in X_ψ such that $X_\psi = \bigcup_{\beta \in B} S_\beta$, and no finite sub-collection covers X . Then $\{S_\beta\}_{\beta \in B} \cup \{\bigcup_{\alpha \in A, \alpha \neq \psi} X_\alpha\}$ is a semi open cover of X with no finite sub-cover. Thus A is finite and (X_α, T_α) is semi compact for all $\alpha \in A$.

Now suppose $A = \{1, 2, \dots, k\}$ and (X_α, T_α) is semi compact for all $\alpha \in A$. Let $\{S_\beta\}_{\beta \in B}$ be a semi open cover of X . Then $\{S_\beta \cap X_\alpha\}_{\beta \in B}$ is a semi open cover of X_α for each $\alpha \in A$, and hence for each $\alpha \in A$ there is a finite sub-collection $C_\alpha \subset \{S_\beta \cap X_\alpha\}_{\beta \in B}$ such that $X_\alpha = \bigcup_{S_\beta \cap X_\alpha \in C_\alpha} (S_\beta \cap X_\alpha)$. Let $S = \{S_\beta \mid S_\beta \cap X_\alpha \in C_\alpha \text{ for some } \alpha \in A\}$. Then S is a finite sub-collection of $\{S_\beta\}_{\beta \in B}$ which covers X , and X is semi compact.

Investigation into the question of which subspaces of semi compact spaces are semi compact led to the inclusion in the characterization theorem of classification of semi compact spaces into types described in the following definition.

Definition 2.1. Let (X, T) be a topological space. Then

- (a) (X, T) is type 1 if and only if X is finite,
- (b) (X, T) is type 2 if and only if X is infinite and for each $0 \in T - \{\emptyset\}$, $X - 0 = \text{Fr}(0)$ is finite, and
- (c) (X, T) is type 3 if and only if (X, T) is not type 2 and there exists a finite sequence $\{0_i\}_{i=1}^n$ of disjoint open sets such that $(0_i, T_{0_i})$ is type 2 for each $i \in \{1, 2, \dots, n\}$, $X - \bigcup_{i=1}^n 0_i$ is finite, and X is connected.

The notion of semi embedding is also used in the characterization theorem.

Definition 2.2. A one-to-one function f from a topological space (X, T) into a topological space (Y, S) is a semi embedding if and only if $f : (X, T) \rightarrow (f(X), S_{f(X)})$ is a semi homeomorphism.

Theorem 2.3. Let (X, T) be a topological space. Then the following are equivalent:

- (a) (X, T) is semi compact,
- (b) for each $Y \in T$, (Y, T_Y) is semi compact,
- (c) for each $0 \in T$, $\text{Fr}(0)$ is finite and if $\{0_n\}_{n \in \mathbb{N}}$ is a sequence of open sets in X , then there exists $m \in \mathbb{N}$ such that $0_p \subset \bigcup_{i=1}^m 0_i$ for all $p \in \mathbb{N}$,
- (d) for each $0 \in T$, $\text{Fr}(0)$ is finite and every chain in T , partially ordered by set containment, has a maximal element,
- (e) for each $0 \in SO(X, T)$, $\text{Fr}(0)$ is finite and every chain in $SO(X, T)$, partially ordered by set containment, has a maximal element,

- (f) for each $0 \in S0(X, T)$, $Fr(0)$ is finite and if $\{0_n\}_{n \in \mathbb{N}}$ is a sequence of semi open sets in X , then there exists $m \in \mathbb{N}$ such that $0_p \subset \bigcup_{i=1}^m 0_i$ for all $p \in \mathbb{N}$,
- (g) for each $Y \subset X$, (Y, T_Y) is semi compact,
- (h) for each $Y \subset X$, every cover of Y by sets semi open in X has a finite sub-cover,
- (i) (X, T) is type 1, type 2, type 3, or the finite topological sum of nonempty disjoint spaces that are type 1, type 2, or type 3,
- (j) every semi homeomorphic image of (X, T) is semi compact, and
- (k) (X, T) can be semi embedded in a semi compact space.

Proof: (a) implies (b). Let $Y \in T$ and let $A \in T_Y$. Then $A \in T$ so that $Fr_X(A)$ is finite. Since $Fr_Y(A) \subset Fr_X(A)$, then $Fr_Y(A)$ is finite. If there exists an infinite sequence of nonempty disjoint open sets in Y , then the same sets form an infinite sequence of nonempty disjoint open sets in X , which is impossible since (X, T) is semi compact.

(b) implies (c). Since $X \in T$, then (X, T) is semi compact and $Fr(0)$ is finite for each $0 \in T$. Let $\{0_n\}_{n \in \mathbb{N}}$ be a sequence of open sets in X . Then $Y = \bigcup_{n \in \mathbb{N}} 0_n \in T$ and thus (Y, T_Y) is semi compact. Since $\{0_n\}_{n \in \mathbb{N}}$ is an open cover of Y , then there exists a finite sub-collection $\{0_{n_i}\}_{i=1}^k$ that covers Y . Let $m = \max \{n_1, n_2, \dots, n_k\}$. Then $Y = \bigcup_{i=1}^m 0_i$, and $0_p \subset \bigcup_{i=1}^m 0_i$ for all $p \in \mathbb{N}$.

(c) implies (d). Suppose there exists a chain $C = \{0_\alpha\}_{\alpha \in A}$ in T partially ordered by set containment with no maximal element. For

all $p \in \mathbb{N}$, there exists $f_p : N_p \rightarrow C$, with $f_p(i) = 0_i$, such that (1) if $i \in N_p - \{1\}$, then f_i is an extension of f_{i-1} , $\bigcup_{j < i} 0_j \subset 0_i$, and $0_i \cap (X - \bigcup_{j < i} 0_j) \neq \emptyset$, and (2) for all $i \in \mathbb{N}$, 0_i is not maximal in C . The proof is by mathematical induction.

Since C has no maximal element, then C contains a nonempty set 0_α that is not maximal. Let $0_1 = 0_\alpha$ and $f_1(1) = 0_1$. Suppose the statement is true for $p = k$. Since 0_k is not maximal in C , then there exists $0_\beta \in C$ such that $0_\beta \not\subset 0_k$. Since C is a chain, then $0_k \subsetneq 0_\beta$. Let $0_{k+1} = 0_\beta$. Since $\bigcup_{j < k} 0_j \subset 0_k$ and $0_k \subset 0_{k+1}$, then $\bigcup_{j < k+1} 0_j \subset 0_{k+1}$. Also, $0_{k+1} \cap (X - \bigcup_{j < k+1} 0_j) \neq \emptyset$. Then $f_{k+1} : N_{k+1} \rightarrow C$ defined by

$$f_{k+1}(i) = \begin{cases} 0_{k+1} & \text{if } i = k+1 \\ f_k(i) & \text{if } i \leq k \end{cases}$$

has the desired properties. Then $\{0_n\}_{n \in \mathbb{N}}$ is a sequence of open sets in X and for each $p \in \mathbb{N}$, $0_p \not\subset \bigcup_{i=1}^m 0_i$ for all $m < p$, which contradicts (c).

(d) implies (e). Let $S \in SO(X, T)$ and let $0 \in T$ such that $0 \subset S \subset \overline{0}$. Then $\text{Fr}(S) \subset \text{Fr}(0)$, which is finite. Let $C = \{S_\alpha\}_{\alpha \in A}$ be a chain in $SO(X, T)$ partially ordered by set containment, and for each $\alpha \in A$ choose $0_\alpha \in T$ such that $0_\alpha = \text{Int}(S_\alpha) \subset S_\alpha \subset \overline{0}_\alpha$. Then $\{0_\alpha\}_{\alpha \in A}$ is a chain in T partially ordered by set containment and hence contains a maximal element 0_β .

For each $j \in A$, $S_j \subset \overline{0}_j \subset \overline{0}_\beta = 0_\beta \cup \text{Fr}(0_\beta)$. Thus for each $j \in A$, $S_j - S_\beta \subset \text{Fr}(0_\beta)$.

Let $F = \{x \in \text{Fr}(0_\beta) \mid x \in S_j \text{ for some } j \in A\}$. Then for each $j \in A$, $S_j \subset 0_\beta \cup F$.

If $F = \emptyset$, $0_\beta = S_\beta$ and $S_j \subset S_\beta$ for all $j \in A$, which implies that C has a maximal element, S_β . Suppose $F \neq \emptyset$. Since $\text{Fr}(0_\beta)$ is finite, $F = \{x_1, x_2, \dots, x_k\}$ for some $k \in \mathbb{N}$. For each $i \in \{1, 2, \dots, k\}$, choose $S_{\alpha_i} \in C$ such that $x_i \in S_{\alpha_i}$. Let $S = \{S_\beta\} \cup \{S_{\alpha_1}, S_{\alpha_2}, \dots, S_{\alpha_k}\}$. Since S is a finite subset of the chain C , then S has a maximal element, S_m , and $S_m \subset 0_\beta \cup F$ since $S_m \in C$. Since $S_\beta \in S$, then $S_\beta \subset S_m$, which implies $0_\beta \subset S_m$. Also, since $F \subset \bigcup_{i=1}^k S_{\alpha_i}$, then $F \subset S_m$. Then $0_\beta \cup F \subset S_m$, and $0_\beta \cup F = S_m$. Hence, C contains a maximal element, S_m .

(e) implies (f). Let $\{0_n\}_{n \in \mathbb{N}}$ be a sequence of semi open sets in X and for each $j \in \mathbb{N}$, let $S_j = \bigcup_{n \leq j} 0_n$. Then $\{S_j\}$ is a chain in $SO(X, T)$. Then $\{S_j\}_{j \in \mathbb{N}}$ has a maximal element, S_m , and $0_p \subset S_p \subset S_m = \bigcup_{i=1}^m 0_i$ for all $p \in \mathbb{N}$.

(f) implies (g). Suppose there exists $Y \subset X$ such that (Y, T_Y) is not semi compact. Then Y is infinite and there is a collection $C = \{S_\alpha\}_{\alpha \in A}$ of sets in $SO(Y, T_Y)$ such that $Y = \bigcup_{\alpha \in A} S_\alpha$ and no finite sub-collection of C covers Y . Thus no finite sub-collection of C contains all but finitely many elements of Y . Then for each $p \in \mathbb{N}$, there exists $f_p : N_p \rightarrow (Y, C, T_Y, T)$, with $f_p(i) = (x_i, S_{\alpha_i}, 0_{\alpha_i}, Q_{\alpha_i})$, such that

- (1) if $i \in N_p - \{1\}$, then f_i is an extension of f_{i-1} and $x_i \in Y - \overline{\bigcup_{k < i} Q_{\alpha_k}}$, and
- (2) for all $i \in N_p$, $x_i \in S_{\alpha_i} \subset \overline{Q_{\alpha_i}}$, $0_{\alpha_i} = Q_{\alpha_i} \cap Y$, $0_{\alpha_i} \subset S_{\alpha_i} \subset \overline{Q_{\alpha_i}}$, and $Y - \overline{\bigcup_{j \leq i} Q_{\alpha_j}}$ is infinite. The proof is by mathematical induction.

Let $x_1 \in Y$ and choose S_{α_1} such that $x_1 \in S_{\alpha_1}$. Then there exists $0_{\alpha_1} \in T_Y$ such that $0_{\alpha_1} \subset S_{\alpha_1} \subset \overline{0_{\alpha_1}}_Y$, and there exists $Q_{\alpha_1} \in T$ such that $0_{\alpha_1} = Y \cap Q_{\alpha_1}$.

Then $S_{\alpha_1} \subset \overline{Q_{\alpha_1}}_X$ and $Y - Q_{\alpha_1}$ is infinite. Since $\text{Fr}_X(Q_{\alpha_1})$ is finite, then

$Y - \overline{Q_{\alpha_1}}_X$ is infinite. Thus $f_1 : N_1 \rightarrow (Y, C, T_Y, T)$ defined by

$f_1(1) = (x_1, S_{\alpha_1}, 0_{\alpha_1}, Q_{\alpha_1})$ satisfies the desired conditions. Suppose

the statement is true for $p = k$. Since $Y - \bigcup_{j \leq k} \overline{Q_{\alpha_j}}$ is infinite, let $x_{k+1} \in$

$Y - \bigcup_{j \leq k} \overline{Q_{\alpha_j}}$. Let $S_{\alpha_{k+1}} \in C$ such that $x_{k+1} \in S_{\alpha_{k+1}}$, let $0_{\alpha_{k+1}} \in T_Y$ such that

$0_{\alpha_{k+1}} \subset S_{\alpha_{k+1}} \subset \overline{0_{\alpha_{k+1}}}_Y$, and let $Q_{\alpha_{k+1}} \in T$ such that $0_{\alpha_{k+1}} = Y \cap Q_{\alpha_{k+1}}$.

Then $Y - \bigcup_{j=1}^{k+1} \overline{Q_{\alpha_j}}$ is infinite, since $\bigcup_{j=1}^{k+1} 0_{\alpha_j} = \bigcup_{j=1}^{k+1} (Q_{\alpha_j} \cap Y)$ and $\bigcup_{j=1}^{k+1} 0_{\alpha_j} \subset \bigcup_{j=1}^{k+1} S_{\alpha_j}$.

Since $\text{Fr}_X(\bigcup_{j=1}^{k+1} Q_{\alpha_j})$ is finite, then $Y - \bigcup_{j \leq k+1} \overline{Q_{\alpha_j}}$ is infinite. Then

$f_{k+1} : N_{k+1} \rightarrow (Y, C, T_Y, T)$ defined by

$$f_{k+1}(i) = \begin{cases} (x_{k+1}, S_{\alpha_{k+1}}, 0_{\alpha_{k+1}}, Q_{\alpha_{k+1}}) & \text{if } i = k+1 \\ f_k(i) & \text{if } i \leq k \end{cases}$$

has the desired properties. Then $\{x_n \mid n \in N\} \subset \bigcup_{n \in N} \overline{Q_{\alpha_n}}$. Since $\{\overline{Q_{\alpha_n}}\}_{n \in N}$

is a sequence in $SO(X, T)$, there exists $n \in N$ such that $\overline{Q_{\alpha_p}} \subset \bigcup_{i=1}^m \overline{Q_{\alpha_i}}$

for all $p \in N$, but $\bigcup_{i=1}^m \overline{Q_{\alpha_i}} = \overline{\bigcup_{i=1}^m Q_{\alpha_i}}$, $x_{m+1} \in \overline{Q_{\alpha_{m+1}}}$, and $x_{m+1} \notin \bigcup_{i=1}^m Q_{\alpha_i}$, which

is impossible. Hence (Y, T_Y) is semi compact.

(g) implies (h). Let $Y \subset X$ and let $\{S_\alpha\}_{\alpha \in A}$ be a cover of Y , where $S \in SO(X, T)$ for all $\alpha \in A$. Let $W = \bigcup_{\alpha \in A} S_\alpha$. Since (W, T_W) is semi compact, and $\{S_\alpha\}_{\alpha \in A}$ is a cover of W by semi open sets in W , then there is a finite sub-cover of $\{S_\alpha\}_{\alpha \in A}$ for W . This same finite sub-cover covers Y .

(h) implies (i). Since $X \subset X$, then (X, T) is semi compact. If X is finite, then (X, T) is type 1 and (i) is true. Suppose X is infinite. Then there exists $Y \in T$ such that Y is infinite and if $V \in T - \{\emptyset\}$ such that $V \subset Y$, then $Y \subset \bar{V}$. Let $\omega = \{W \in T \mid Y \subset W \text{ and if } V \in T - \{\emptyset\} \text{ such that } V \subset W, \text{ then } W \subset \bar{V}\}$. Then ω is not empty since $Y \in \omega$. Partial order ω by set containment. Since (X, T) is semi compact, then, by (d) above, every chain in T has a maximal element. Thus, by Zorn's Lemma, ω has a maximal element M_Y . Let $u = \{U \in T \mid U \text{ is infinite and if } V \in T - \{\emptyset\} \text{ such that } V \subset U, \text{ then } U \subset \bar{V}\}$, and let $M = \{M_U \mid U \in u \text{ and } M_U \text{ is the maximal element of } u \text{ containing } U\}$. Then M_U is connected for all $M_U \in M$. For each $M_U \in M$ let C_U be the component of X containing M_U .

Suppose there exist M_R and $M_S \in M$ such that $M_R \cap M_S \neq \emptyset$. Then $R \subset M_R \subset M_R \cup M_S$. Let $V \in T - \{\emptyset\}$ such that $V \subset M_R \cup M_S$. Suppose $V \cap M_R \neq \emptyset$. Then $V \cap M_R \in T - \{\emptyset\}$ and $V \cap M_R \subset M_R$, which implies $M_R \subset \overline{V \cap M_R} \subset \bar{V}$. Also, $V \cap M_S \in T - \{\emptyset\}$ and $V \cap M_S \subset M_S$, which implies $M_S \subset \overline{V \cap M_S} \subset \bar{V}$. Hence, $M_R \cup M_S \subset \bar{V}$. Suppose $V \cap M_S = \emptyset$. Then $V \subset M_R$, which implies $M_R \subset \bar{V}$. Since $M_R \cap M_S \in T - \{\emptyset\}$, and $M_R \cap M_S \subset M_S$, then $M_S \subset \overline{M_R \cap M_S} \subset \bar{M_R} \subset \bar{V}$. Hence, $M_R \cup M_S \subset \bar{V}$. Similarly, if $V \cap M_R = \emptyset$, then $M_R \cup M_S \subset \bar{V}$. Since M_R is maximal with respect to the properties that $M_R \cup M_S$ possesses, then $M_S \subset M_R$. Similarly, $M_R \subset M_S$. Hence, M is a collection of disjoint sets.

Then M is finite, since otherwise M is an infinite collection of nonempty disjoint open sets, which is impossible. Hence,

$C = \{C_U \mid U \in M\}$ is finite.

Let $C = \{C_{U_i}\}_{i=1}^n$, and let $F = X - \bigcup_{i=1}^n C_{U_i}$. Suppose F is infinite.

Since (F, T_F) is semi compact, then F contains $Y \in T$ such that Y is infinite and if $V \in T - \{\emptyset\}$ such that $V \subset Y$, then $Y \subset \bar{V}$. Then $Y \subset M_Y \in M$, and $Y \subset C_{U_i}$ for some $i \in \{1, 2, \dots, n\}$, which contradicts the fact that $Y \subset F$.

Hence, F is finite. Then F is open since each component is closed.

Let $F = \{x_1, x_2, \dots, x_k\}$. Then $\bar{F} = \{\bar{x}_1\} \cup \{\bar{x}_2\} \cup \dots \cup \{\bar{x}_k\}$. Suppose

there exists $j \in \{1, 2, \dots, k\}$ such that $\{\bar{x}_j\} \cap (\bigcup_{i=1}^n C_{U_i}) \neq \emptyset$. Then there

exists $p \in \{1, 2, \dots, n\}$ such that $\{\bar{x}_j\} \cap C_{U_p} \neq \emptyset$. Since $\{x_j\}$ is connected,

then $\{\bar{x}_j\}$ is connected. Hence, $\{\bar{x}_j\} \cup C_{U_p}$ is closed and connected,

which contradicts the fact that C_{U_p} is a component of X . Hence,

$\bar{F} \subset F$, and F is closed. Then $X = C_{U_1} \cup C_{U_2} \cup \dots \cup C_{U_n} \cup F$, which is the

union of a finite number of disjoint subsets, each of which is open and closed.

Since F is finite, F is type 1.

Let $M \in \mathcal{M}$. By definition of M , M is infinite. Let $0 \in T_M$. Since, by part (b) above, (M, T_M) is semi compact, then, by part (c) above, $\text{Fr}_M(0)$ is finite. Since $0 \in T_M$, then $\text{Fr}_M(0) \subset M - 0$. Since $M \subset \bar{0}_M = 0 \cup \text{Fr}_M(0)$, then $M - 0 \subset \text{Fr}_M(0)$. Hence, (M, T_M) is type 2 for each $M \in \mathcal{M}$.

Let $D \in \mathcal{C}$. By definition of $\mathcal{C}$, D contains M for some $M \in \mathcal{M}$. If $D = M$, then (D, T_D) is type 2.

Consider the case that $D \neq M$ and $D-M$ is finite. If $D-M \subset \text{Fr}_D(M)$, then (D, T_D) is type 2. If $D-M \not\subset \text{Fr}_D(M)$, then (D, T_D) satisfies the definition for type 3 with $n = 1$.

Now consider the case that $D \neq M$ and $D-M$ is infinite. Since $M \in T_D$, then (D, T_D) is not type 2. Since $D-M$ is infinite and (D, T_D) is semi compact, then there exists $U \in \mathcal{U}$, and consequently $M_U \in M$, such that $U \subset M_U \subset D-M$. Since M is finite, then D contains at most a finite number of sets in M . Also, $D - \bigcup_{P \in M, P \subset D} P$ is finite, since otherwise

$D - \bigcup_{P \in M, P \subset D} P$ contains $V \in \mathcal{U}$ and consequently $M_V \in M$, which is impossible.

Since M is a disjoint collection of open sets, for each $M \in M$, (M, T_M) is type 2, and since D is connected, then (D, T_D) is type 3.

Hence (X, T) is type 1, type 2, type 3, or a finite topological sum of nonempty disjoint spaces that are type 1, type 2, or type 3.

(i) implies (j). If (X, T) is type 1, then X is finite and hence semi compact.

If (X, T) is type 2, then $\text{Fr}(0)$ is finite for all $0 \in T$ and there does not exist an infinite sequence of nonempty disjoint open sets, which implies (X, T) is semi compact.

Suppose (X, T) is type 3. Then there exists a finite sequence $\{0_i\}_{i=1}^n$ of disjoint open sets such that $(0_i, T_{0_i})$ is type 2 for each $i \in \{1, 2, \dots, n\}$, and $X - \bigcup_{i=1}^n 0_i$ is finite. Let $\{S_\alpha\}_{\alpha \in A}$ be a semi open cover of X . Then $\{0_i \cap S_\alpha\}_{\alpha \in A}$ is a semi open cover of 0_i for each $i \in \{1, 2, \dots, n\}$. Since $(0_i, T_{0_i})$ is type 2 for each $i \in \{1, 2, \dots, n\}$, then $(0_i, T_{0_i})$ is semi compact for each $i \in \{1, 2, \dots, n\}$. For each $i \in \{1, 2, \dots, n\}$, let A_i be a finite sub-collection of $\{0_i \cap S_\alpha\}_{\alpha \in A}$ that covers 0_i , and let $S_i = \{S_\beta \mid 0_i \cap S_\beta \in A_i\}$.

For each $x \in X - \bigcup_{i=1}^n O_i$, let $S_{\alpha_x} \in \{S_\alpha\}_{\alpha \in A}$ such that $x \in S_{\alpha_x}$. Then

$(\bigcup_{i=1}^n S_i) \cup \{S_{\alpha_x} \mid x \in X - \bigcup_{i=1}^n O_i\}$ is a finite sub-collection of $\{S_\alpha\}_{\alpha \in A}$ that

covers X . Thus (X, T) is semi compact.

If (X, T) is a finite topological sum of disjoint spaces that are type 1, type 2, or type 3, then, since each space is semi compact, (X, T) is semi compact.

Let $f : (X, T) \rightarrow (Y, S)$ be a semi homeomorphism, and let $\{S_\alpha\}_{\alpha \in A}$ be a semi open cover of Y . Then $\{f^{-1}(S_\alpha)\}_{\alpha \in A}$ is a semi open cover of X . Since (X, T) is semi compact, then there exists a finite sub-collection $\{f^{-1}(S_{\alpha_i})\}_{i=1}^n$ of $\{f^{-1}(S_\alpha)\}_{\alpha \in A}$ that covers X . Then $\{S_{\alpha_i}\}_{i=1}^n$ is a finite sub-collection of $\{S_\alpha\}_{\alpha \in A}$ that covers Y , and (Y, S) is semi compact.

(j) implies (k). Let $f : (X, T) \rightarrow (X, T)$ be defined by $f(x) = x$. Then f is a semi embedding of (X, T) into a semi compact space.

(k) implies (a). Let (Y, S) be a semi compact space such that (X, T) can be semi embedded in (Y, S) , and let f be a semi embedding of (X, T) into (Y, S) . Since (Y, S) is semi compact, then part (g) above implies $(f(X), S_{f(X)})$ is semi compact, and part (j) above implies every semi homeomorphic image of $(f(X), S_{f(X)})$ is semi compact. Then since $f^{-1} : (f(X), S_{f(X)}) \rightarrow (X, T)$ is a semi homeomorphism, then (X, T) is semi compact.

CHAPTER III

SEMI COMPACT T_0 SPACES, T_1 SPACES, AND R_0 SPACES

Since the time that Dorsett found equivalent conditions for semi compact T_2 spaces and R_1 spaces, the desire grew to describe equivalent conditions for semi compactness in T_0 spaces, T_1 spaces, and R_0 spaces.

The following observation combined with the characterization theorem of Chapter II yields the desired description for T_0 spaces.

Theorem 3.1. Let $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$ be a family of nonempty disjoint topological spaces, and let (X, T) be the topological sum of $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$. Then (X, T) is T_0 if and only if (X_α, T_α) is T_0 for all $\alpha \in A$, and (X, T) is T_1 if and only if (X_α, T_α) is T_1 for all $\alpha \in A$.

The straightforward proof is omitted.

Corollary 3.1. A topological space (X, T) is semi compact T_0 if and only if (X, T) is a T_0 type 1 space, a T_0 type 2 space, a T_0 type 3 space, or a finite topological sum of disjoint spaces that are T_0 type 1, T_0 type 2, or T_0 type 3.

Classifying semi compact spaces into type 1, type 2, or type 3 motivated the idea of a similar classification into types for semi compact T_1 spaces. These types are described in the following definition.

Definition 3.1. Let (X, T) be a topological space. Then

- (a) (X, T) is type (1,1) if and only if (X, T) is T_2 and X is finite,
- (b) (X, T) is type (2,1) if and only if X is infinite and T is the finite complement topology on X , and

- (c) (X, T) is type $(3, 1)$ if and only if there exists a finite sequence $\{O_i\}_{i=1}^n$, $n \geq 2$, of disjoint open sets such that (O_i, T_{O_i}) is type $(2, 1)$ for each $i \in \{1, 2, \dots, n\}$, $X - \bigcup_{i=1}^n O_i = \text{Fr}(\bigcup_{i=1}^n O_i)$ is finite, and (X, T) is connected T_1 .

The following theorem, together with the characterization theorem in Chapter II and the observation in Theorem 3.1, yields the desired classification for semi compact T_1 spaces given in Corollary 3.2.

Theorem 3.2. Let (X, T) be a topological space. Then

- (a) (X, T) is type $(1, 1)$ if and only if it is T_1 type 1,
- (b) (X, T) is type $(2, 1)$ if and only if it is T_1 type 2, and
- (c) (X, T) is type $(3, 1)$ if and only if it is T_1 type 3.

Proof: If (X, T) is type $(1, 1)$, then X is finite and (X, T) is T_2 , which implies (X, T) is T_1 type 1. If (X, T) is T_1 type 1, then X is finite and T is the discrete topology on X , which implies (X, T) is type $(1, 1)$.

If (X, T) is type $(2, 1)$, then X is infinite and T is the finite complement topology on X , which implies (X, T) is T_1 type 2. Suppose (X, T) is T_1 type 2. Then X is infinite, (X, T) is T_1 , and $X - O = \text{Fr}(O)$ is finite for each $O \in T - \{\emptyset\}$. Let $U \subset X$ such that $X - U$ is finite. For each $x \in X - U$, $\{x\}$ is closed, since (X, T) is T_1 . Then $X - U$ is closed since the finite union of closed sets is closed, which implies $U \in T$. Then T is the finite complement topology on X . Thus (X, T) is type $(2, 1)$.

Now suppose (X, T) is type $(3, 1)$. Then there exists a finite sequence $\{O_i\}_{i=1}^n$, $n \geq 2$, of disjoint open sets such that (O_i, T_{O_i}) is type $(2, 1)$ for each $i \in \{1, 2, \dots, n\}$, $X - \bigcup_{i=1}^n O_i = \text{Fr}(\bigcup_{i=1}^n O_i)$ is finite, and (X, T) is connected T_1 . Then (O_i, T_{O_i}) is type 2 for each $i \in \{1, 2, \dots, n\}$,

and since $n \geq 2$, then (X, T) is not type 2. Thus (X, T) satisfies the conditions for type 3.

Now suppose (X, T) is T_1 type 3. Then (X, T) is not type 2, and there exists a finite sequence $\{O_i\}_{i=1}^n$ of disjoint open sets such that (O_i, T_{O_i}) is type 2 for each $i \in \{1, 2, \dots, n\}$, $X - \bigcup_{i=1}^n O_i$ is finite, and X is connected. Suppose $n = 1$. Since (O_1, T_{O_1}) is type 2, then O_1 is infinite and hence X is infinite. Let $U \in T - \{\emptyset\}$. Then $X - U = (O_1 - U) \cup ((X - O_1) - U)$, which is finite since $O_1 - U$ and $X - O_1$ are finite. Then $X - \bar{U}$ is finite. Since (X, T) is T_1 , then $X - \bar{U}$ is closed. Hence, $\bar{U}$ is open. Since $X = \bar{U} \cup (X - \bar{U})$, $\bar{U}$ and $X - \bar{U}$ are open and disjoint, X is connected, and $\bar{U} \neq \emptyset$, then $X - \bar{U} = \emptyset$. Thus $X = \bar{U} = U \cup \text{Fr}(U)$. Since $U \in T - \{\emptyset\}$, then $U \cap \text{Fr}(U) = \emptyset$. Hence $\text{Fr}(U) = X - U$, which is finite. Then (X, T) is type 2, which is a contradiction. Hence, $n \geq 2$. Since every subspace of a T_1 space is T_1 , then (O_i, T_{O_i}) is T_1 for each $i \in \{1, 2, \dots, n\}$. Also, $\text{Fr}(\bigcup_{i=1}^n O_i) = X - \bigcup_{i=1}^n O_i$ since otherwise, $\bigcup_{i=1}^n \bar{O}_i$ and the finite set $X - \bigcup_{i=1}^n \bar{O}_i$ would be two nonempty disjoint closed sets whose union is X , which contradicts the fact that X is connected. Hence, (X, T) is type (3,1).

Corollary 3.2. Let (X, T) be a topological space. Then (X, T) is semi compact T_1 if and only if (X, T) is type (1,1), type (2,1), type (3,1), or a finite topological sum of nonempty disjoint spaces that are type (1,1), type (2,1), or type (3,1).

The types for semi compact R_0 spaces are described in the following definition.

Definition 3.3. Let (X, T) be a topological space. Then

- (a) (X, T) is type $(1, 0)$ if and only if $\{\{\bar{x}\} | x \in X\}$ is a finite decomposition of X ,
- (b) (X, T) is type $(2, 0)$ if and only if $\text{Fr}(0)$ is finite for all $0 \in T$ and $(X_0, Q(X_0))$ is type $(2, 1)$, and
- (c) (X, T) is type $(3, 0)$ if and only if there exists a finite sequence $\{O_i\}_{i=1}^n$, $n \geq 2$, of disjoint open sets such that (O_i, T_{O_i}) is type $(2, 0)$ for all $i \in \{1, 2, \dots, n\}$, $X - \bigcup_{i=1}^n O_i = \text{Fr}(\bigcup_{i=1}^n O_i)$ is finite, and (X, T) is connected R_0 .

The following theorem characterizes semi compact R_0 spaces in terms of these types.

Theorem 3.3. Let (X, T) be a topological space. Then the following are equivalent:

- (a) (X, T) is semi compact R_0 ,
- (b) $(X_0, Q(X_0))$ is semi compact T_1 and for each closed set C in X_0 with empty interior, each element of C is a finite set, and
- (c) (X, T) is type $(1, 0)$, type $(2, 0)$, type $(3, 0)$, or a finite topological sum of nonempty disjoint spaces that are type $(1, 0)$, type $(2, 0)$, or type $(3, 0)$.

Proof: (a) implies (b). Since (X, T) is R_0 , then $(X_0, Q(X_0))$ is T_1 . Since the natural map $P : (X, T) \rightarrow (X_0, Q(X_0))$ is continuous, open, and onto, then $(X_0, Q(X_0))$ is semi compact. Now let C be closed in $(X_0, Q(X_0))$ such that the interior of C is empty. Let $Y \in C$. Since (X, T) is R_0 , then $Y = \{\bar{x}\}$ for some $x \in X$. Then $\text{Int}(\{\bar{x}\}) = \emptyset$, since if $0 \in T - \{\emptyset\}$ such that $0 \subset \{\bar{x}\}$, $P(0) = \{\{\bar{x}\}\} \in Q(X_0)$ and $\{\{\bar{x}\}\} \subset C$, which is impossible. Thus $\text{Int}(\{\bar{x}\})$ is empty, which implies $Y = \{\bar{x}\}$ is finite.

(b) implies (c). By Corollary 3.2, $(X_0, Q(X_0))$ is type (1,1), type (2,1), type (3,1), or a finite topological sum of nonempty disjoint spaces that are type (1,1), type (2,1), or type (3,1). Since $(X_0, Q(X_0))$ is T_1 , then $\{\{\bar{x}\} | x \in X\}$ is a decomposition of X .

If $(X_0, Q(X_0))$ is type (1,1), then X_0 is finite and (X, T) is type (1,0).

Suppose $(X_0, Q(X_0))$ is type (2,1). Then X_0 is infinite and $Q(X_0)$ is the finite complement topology on X_0 . Let $0 \in T$. Then $P(0) \in Q(X_0)$, and $P^{-1}(\text{Fr}(P(0))) = \text{Fr}(P^{-1}(P(0))) = \text{Fr}(0)$. Since $\text{Fr}(P(0))$ is a closed set with empty interior, then $\{\bar{x}\}$ is finite for each $\{\bar{x}\} \in \text{Fr}(P(0))$, and since $\text{Fr}(P(0))$ is finite, then $P^{-1}(\text{Fr}(P(0)))$ is the finite union of finite sets and hence finite. Thus (X, T) is type (2,0).

Now suppose $(X_0, Q(X_0))$ is type (3,1). Then there exists a finite sequence $\{0_i\}_{i=1}^n$, $n \geq 2$, of disjoint open sets such that $(0_i, Q(X_0)_{0_i})$ is type (2,1) for each $i \in \{1, 2, \dots, n\}$, $X_0 - \bigcup_{i=1}^n 0_i = \text{Fr}(\bigcup_{i=1}^n 0_i)$ is finite, and $(X_0, Q(X_0))$ is connected T_1 . Since $(X_0, Q(X_0))$ is T_1 , then (X, T) is R_0 ; since X_0 is connected, then $P^{-1}(X_0) = X$ is connected; and since $\{0_i\}_{i=1}^n$ is a disjoint sequence of open sets, then $\{P^{-1}(0_i)\}_{i=1}^n$ is a disjoint sequence of open sets. Also, since $(0_i, T_{0_i})$ is type (2,1) for each $i \in \{1, 2, \dots, n\}$, then by the argument above $(P^{-1}(0_i), T_{P^{-1}(0_i)})$ is type (2,0) for each $i \in \{1, 2, \dots, n\}$ and $\text{Fr}(\bigcup_{i=1}^n P^{-1}(0_i)) = \text{Fr}(P^{-1}(\bigcup_{i=1}^n 0_i)) = P^{-1}(\text{Fr}(\bigcup_{i=1}^n 0_i)) = P^{-1}(X_0 - \bigcup_{i=1}^n 0_i) = X - P^{-1}(\bigcup_{i=1}^n 0_i) = X - \bigcup_{i=1}^n P^{-1}(0_i)$, which is finite. Hence, (X, T) is type (3,0).

(c) implies (a). Suppose (X, T) is type $(1, 0)$. Then $\{\overline{\{x\}} \mid x \in X\}$ is a finite decomposition of X and (X, T) is R_0 . Let $X = \overline{\{x_1\}} \cup \overline{\{x_2\}} \cup \dots \cup \overline{\{x_k\}}$. Since $\overline{\{x_i\}}$ is closed for each $i \in \{1, 2, \dots, k\}$, and $\overline{\{x_i\}} = X - \bigcup_{j \in \{1, 2, \dots, k\}, j \neq i} \overline{\{x_j\}}$, then $\overline{\{x_i\}}$ is open for each $i \in \{1, 2, \dots, k\}$. Let $S = \{S_\alpha\}_{\alpha \in A}$ be a semi open cover of X . Let $S_\beta \in S$, and let $0_\beta \in T$ such that $0_\beta \subset S_\beta \subset \overline{0_\beta}$. Suppose there exists $x \in X$ such that $x \in \text{Fr}(S_\beta)$. Then $x \in \text{Fr}(0_\beta)$, and $x \in \overline{\{x_j\}}$ for some $j \in \{1, 2, \dots, k\}$. Since $\overline{\{x_j\}}$ is open and $x \in \text{Fr}(0_\beta)$, there exists $y \in 0_\beta$ such that $y \in \overline{\{x_j\}}$. Then $\overline{\{y\}} = \overline{\{x_j\}}$. Since (X, T) is R_0 and $y \in 0_\beta$, then $\overline{\{y\}} \subset 0_\beta$, and hence $\overline{\{x_j\}} \subset 0_\beta$. Hence, $x \in 0_\beta$, which is impossible. Thus $\text{Fr}(S_\alpha) = \emptyset$ for each $\alpha \in A$. For each $i \in \{1, 2, \dots, k\}$ choose $S_{\alpha_i} \in S$ such that $x_i \in S_{\alpha_i}$. Since S_{α_i} is open for each $i \in \{1, 2, \dots, k\}$, then for each $i \in \{1, 2, \dots, k\}$, $\overline{\{x_i\}} \subset S_{\alpha_i}$. Hence, $\{S_{\alpha_i}\}_{i=1}^k$ is a finite cover of X and (X, T) is semi compact R_0 .

Suppose (X, T) is type $(2, 0)$. Then $Q(X_0)$ is the finite complement topology on X_0 , which implies (X, T) is R_0 , and $\text{Fr}(0)$ is finite for each $0 \in T$. Let $S = \{S_\alpha\}_{\alpha \in A}$ be a semi open cover of X . Choose $S_\beta \in S$ and $0 \in T$ such that $0 \subset S_\beta \subset \overline{0}$. Since $X - 0 = P^{-1}(X_0 - P(0)) = P^{-1}(\text{Fr}(P(0))) = \text{Fr}(P^{-1}(P(0))) = \text{Fr}(0)$, then $X - 0$ is finite. For $x \in X - 0$, choose $S_{\alpha_x} \in S$ such that $x \in S_{\alpha_x}$. Then $\{S_\beta\} \cup \{S_{\alpha_x}\}_{x \in X - 0}$ is a finite sub-cover of X , and (X, T) is semi compact R_0 .

Suppose (X, T) is type $(3, 0)$. Then there exists a finite sequence $\{0_i\}_{i=1}^n$, $n \geq 2$, of disjoint open sets such that $(0_i, T_{0_i})$ is type $(2, 0)$

for all $i \in \{1, 2, \dots, n\}$, $X - \bigcup_{i=1}^n O_i = \text{Fr}(\bigcup_{i=1}^n O_i)$ is finite, and (X, T) is connected R_0 . Let $S = \{S_\alpha\}_{\alpha \in A}$ be a semi open cover of X . Then $\{O_i \cap S_\alpha\}_{\alpha \in A}$ is a semi open cover of O_i in O_i for each $i \in \{1, 2, \dots, n\}$, and there exists a finite sub-collection $C_i \subset \{O_i \cap S_\alpha\}_{\alpha \in A}$ that covers O_i for each $i \in \{1, 2, \dots, n\}$. For each $i \in \{1, 2, \dots, n\}$, let $C_i = \{O_i \cap S_{\alpha_1}, O_i \cap S_{\alpha_2}, \dots, O_i \cap S_{\alpha_{j_i}}\}$, and let $S_i = \{S_{\alpha_1}, S_{\alpha_2}, \dots, S_{\alpha_{j_i}}\}$. Then $\bigcup_{i=1}^n O_i \subset \bigcup_{i=1}^n \bigcup_{m=1}^{j_i} S_{\alpha_m}$. For each $x \in \text{Fr}(\bigcup_{i=1}^n O_i)$, choose $S_{\alpha_x} \in S$ such that $x \in S_{\alpha_x}$. Then $\bigcup_{i=1}^n \{S_{\alpha_m}\}_{m=1}^{j_i} \cup \{S_{\alpha_x}\}_{x \in \text{Fr}(\bigcup_{i=1}^n O_i)}$ is a finite sub-collection of S that covers X , and (X, T) is semi compact R_0 .

Finally, if (X, T) is a finite topological sum of disjoint spaces that are type $(1, 0)$, type $(2, 0)$, or type $(3, 0)$, then (X, T) is a finite topological sum of disjoint spaces that are semi compact R_0 , and (X, T) is semi compact R_0 .

In semi compact T_1 spaces and R_0 spaces, 1st countability and 2nd countability are equivalent, as is shown by the remaining theorems in this chapter.

Theorem 3.4. A topological space (X, T) is 1st countable if and only if for each $Y \in T$, (Y, T_Y) is 1st countable.

The proof is straightforward and is omitted.

Theorem 3.5. Let (X, T) be a semi compact T_1 space. Then the following are equivalent:

- (a) (X, T) is 1st countable,
- (b) X is countable,

- (c) T is countable, and
 (d) (X, T) is 2nd countable.

Proof: (a) implies (b). By Corollary 3.2, (X, T) is type (1,1), type (2,1), type (3,1), or a finite topological sum of disjoint spaces that are type (1,1), type (2,1), or type (3,1).

If (X, T) is type (1,1), then X is finite and hence countable.

Consider the case that (X, T) is type (2,1). Then X is infinite and T is the finite complement topology on X . Let $x \in X$ and let $\{O_i\}_{i \in \mathbb{N}}$ be a countable open neighborhood base at x . Since $X - O_i$ is finite for each $i \in \mathbb{N}$ and $X - \bigcap_{i \in \mathbb{N}} O_i = \bigcup_{i \in \mathbb{N}} (X - O_i)$, then $X - \bigcap_{i \in \mathbb{N}} O_i$ is countable. Suppose $\bigcap_{i \in \mathbb{N}} O_i \neq \{x\}$. Then there exists $y \in X$, $y \neq x$, such that $y \in O_i$ for each $i \in \mathbb{N}$; but $X - \{y\}$ is an open set containing x and hence must contain O_i for some $i \in \mathbb{N}$, which is impossible.

Thus X is countable.

If (X, T) is type (3,1) or a finite topological sum of disjoint spaces that are type (1,1), type (2,1), or type (3,1), then X is countable since it is the finite union of countable sets.

(b) implies (c). By Corollary 3.2, (X, T) is type (1,1), type (2,1), type (3,1), or a finite topological sum of disjoint spaces that are type (1,1), type (2,1), or type (3,1).

If (X, T) is type (1,1), then X is finite, T is finite, and T is countable.

Suppose (X, T) is type (2,1). Then T is the finite complement topology on X . Since the collection of finite subsets of a countable set is countable, then T is countable.

Now suppose (X, T) is type $(3, 1)$. Then there exists a finite sequence $\{O_i\}_{i=1}^n$, $n \geq 2$, of disjoint open sets such that (O_i, T_{O_i}) is type $(2, 1)$ for each $i \in \{1, 2, \dots, n\}$, and $X - \bigcup_{i=1}^n O_i = \text{Fr}(\bigcup_{i=1}^n O_i)$ is finite. Since (O_i, T_{O_i}) is type $(2, 1)$ for each $i \in \{1, 2, \dots, n\}$, then T_{O_i} is countable for each $i \in \{1, 2, \dots, n\}$, and $T \bigcup_{i=1}^n O_i$ is countable. Let $x \in \text{Fr}(\bigcup_{i=1}^n O_i)$ and let $O \in T$ such that $x \in O$. Let A be the collection of subsets of $\text{Fr}(\bigcup_{i=1}^n O_i)$. Then A is finite, and $O = (O \cap (\bigcup_{i=1}^n O_i)) \cup S$, where $S \in A$. Thus each open set containing x can be written as the union of a member of the countable collection $T \bigcup_{i=1}^n O_i$ and a member of the finite set A , which implies the collection of open sets containing x is countable. Therefore, T is countable.

Clearly (c) implies (d) and (d) implies (a).

Theorem 3.6: A topological space is 1st countable if and only if its T_0 -identification space is 1st countable.

Proof: Suppose (X, T) is 1st countable. Let $P : (X, T) \rightarrow (X_0, Q(X_0))$ denote the natural map, let $Y \in X_0$, and let $y \in P^{-1}(Y)$. Then there is a countable open neighborhood base $\{B_i\}_{i \in \mathbb{N}}$ at y . Since P is open, then $\{P(B_i)\}_{i \in \mathbb{N}} \subset Q(X_0)$. Also, $Y \in P(B_i)$ for all $i \in \mathbb{N}$. Let $O \in Q(X_0)$ such that $Y \in O$. Then $y \in P^{-1}(O)$, which is open in (X, T) , which implies there exists $j \in \mathbb{N}$ such that $y \in B_j \subset P^{-1}(O)$. Then $P^{-1}(Y) \subset B_j$, and hence $Y \in P(B_j) \subset O$. Thus the collection $\{P(B_i)\}_{i \in \mathbb{N}}$ forms a countable open neighborhood base at Y , and $(X_0, Q(X_0))$ is 1st countable.

Now suppose $(X_0, Q(X_0))$ is 1st countable. Let $x \in X$. Then $P(x) \in X_0$, and there is a countable open neighborhood base $\{B_i\}_{i \in \mathbb{N}}$ at $P(x)$.

Since P is continuous, then $\{P^{-1}(B_i)\}_{i \in \mathbb{N}} \subset T$. Let $0 \in T$ such that $x \in 0$. Then $P(x) \in P(0)$, and, since P is open, then $P(0) \in Q(X_0)$. Hence there exists $j \in \mathbb{N}$ such that $P(x) \in B_j \subset P(0)$. Then $x \in P^{-1}(P(x)) \subset P^{-1}(B_j) \subset 0$. Therefore $\{P^{-1}(B_i)\}_{i \in \mathbb{N}}$ forms a countable open neighborhood base at x , and (X, T) is 1st countable.

Theorems 1.2, 1.3, 1.4, 1.7, 3.5, and 3.6 may be combined to give the following result.

Corollary 3.4. If (X, T) is semi compact R_0 , then (X, T) is 1st countable if and only if (X, T) is 2nd countable.

It is felt that the formulations given in the characterization theorem, Theorem 2.3, completely describe semi compactness in topological spaces and that semi compact T_0 , R_0 , and T_1 spaces are completely described by Corollary 3.1, Corollary 3.2, and Theorem 3.3 of this paper.

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